FINITE MEAN OSCILLATION AND THE BELTRAMI EQUATION

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ABSTRACT

We prove the existence and uniqueness of homeomorphic ACL solutions to the Beltrami equation in the case when the dilatation coefficient of the equation has a majorant of finite mean oscillation.

1. Introduction

Let D be a domain in $\mathbb{C}$ and μ: $D \to \mathbb{C}$ a measurable function with $|\mu(z)| < 1$ a.e. in D. The **Beltrami equation** associated with μ has the form

$$f_{\overline{z}} = \mu(z) \cdot f_z, \tag{1.1}$$

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where $f_{\bar{z}}$ and f_z are the complex partial derivatives of f. The function μ is the **complex coefficient** of the equation and

$$K(z) = K_\mu(z) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{1.2}$$

is the **dilatation** of the equation. By a **solution** we mean an **ACL solution,** a.e. a continuous mapping $f\colon D \to \mathbb{C}$ which is ACL, i.e. absolutely continuous on lines, cf. [LV], whose partial derivatives, which exist a.e., satisfy (1.1) a.e. If f is a solution, then μ is the **complex dilatation** of f and $K(z)$ is the **dilatation** of f; see [LV]. If $f\colon D \to \mathbb{C}$ is an ACL homeomorphic solution and $K(z) \le Q(z)$ a.e. in D, we say that f is a $Q(z)$-quasiconformal in D.

By the classical existence and uniqueness theorem (see, for instance, [LV] or [Ah]), if $||\mu||_\infty < 1$ or equivalently if $K_\mu \in L^\infty(D)$, then (1.1) has a homeomorphic solution which is unique up to a post composition by a conformal mapping.

Much research has been devoted to the extension of the classical existence and uniqueness theorem to the degenerate case when $||\mu||_\infty = 1$; see, for instance, the monograph [IM] by Iwaniec and Martin and the exposition [SY] by Srebro and Yakubov. In one of the outstanding papers in this direction, David [Da] showed that if $\mu\colon \mathbb{C} \to \mathbb{C}$ is measurable, $|\mu| < 1$ a.e. and satisfies the exponential measure constraint

$$|\{z \in \mathbb{C}\colon |\mu(z)| > 1 - \varepsilon\}| < Ce^{-d/\varepsilon} \tag{1.3}$$

for all $\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0, C > 0$ and $d > 0$, then (1.1) has a unique solution f which maps $\mathbb{C}$ homeomorphically onto itself and fixes the points $0, 1$ and ∞. He also showed that $f \in W^{1,p}_{loc}$ for all $p < 2$, and that $f^{-1} \in W^{1,2}_{loc}$. Tukia [Tu], who extended David's theorem, considered a domain D in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and replaced (1.3) by the spherical exponential condition

$$\sigma\{z \in \mathbb{C}\colon K_\mu(z) > t\} < Ce^{-\alpha t} \tag{1.4}$$

for all $t > T$, for some $C > 0$ and $\alpha > 0$. Here σ denotes the spherical area.

By John-Nirenberg's lemma on BMO functions, (1.3) says that $K_\mu \in BMO(\mathbb{C})$ and (1.4) says that K_μ is BMO in $D, D \subset \overline{\mathbb{C}}$ with respect to the spherical area in $\overline{\mathbb{C}}$.

In [RSY$_2$] we reformulated the theorems of David and Tukia, and showed that if $K_\mu(z) \le Q(z)$ a.e. in D for some $Q \in BMO_{loc}(D)$ then (1.1) has a homeomorphic solution f in D.

For properties of mappings with dilatation majorized by a BMO function we refer to [Da], [Tu], the monograph [IM], the expositions [SY] and [And] and the papers [AIKM], [AS], [IKM], [RSY$_2$] and [Sa].

The main goal in this paper (see Section 4 below) is to show that existence holds if the condition $Q \in BMO_{loc}(D)$ is replaced by the weaker condition that Q is of **finite mean oscillation** in $D, Q \in FMO(D)$. The class of FMO functions which was introduced by Ignat'ev and Ryazanov [IR$_2$] (see Section 2 below for the definition) contains BMO_{loc} as a proper subclass in the strong sense that while $BMO_{loc} \subset L^p_{loc}$ for all $p < \infty$, FMO contains functions which do not belong to any $L^p_{loc}, p > 1$.

As in [RSY$_1$] and as in Brakalova and Jenkins [BJ] who extended David's theorem, our proof of existence is based on approximation and extremal length methods, where the latter is mainly used for equicontinuity; see Section 3 below.

From the well-known factorization theorem of Iwaniec and Švérak, Theorem 1 in [IS], one obtains that if $f\colon D \to \mathbb{C}$ is a homeomorphism, $f \in W^{1,2}_{loc}(D), J(z,f) \geq 0$ a.e. and $K_f \in L^1_{loc}(D)$, then $f^{-1} \in W^{1,2}_{loc}(D)$. However, as in several other existence theorems, the homeomorphic solution f which is constructed here need not belong to $W^{1,2}_{loc}$. Nevertheless, $f^{-1} \in W^{1,2}_{loc}$, which in turn implies that f generates all $W^{1,2}_{loc}$ solutions by means of compositions with holomorphic mappings; see Section 5.

2. Finite mean oscillation

Let D be a domain in the complex plane $\mathbb{C}$. We say that a function $\varphi\colon D \to \mathbb{R}$ has **finite mean oscillation** at a point $z_0 \in D$ if

$$d_\varphi(z_0) = \overline{\lim_{\varepsilon \to 0}} \fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)| dxdy < \infty, \tag{2.1}$$

where

$$\overline{\varphi}_\varepsilon(z_0) = \fint_{D(z_0,\varepsilon)} \varphi(z) dxdy \tag{2.2}$$

is the mean value of the function $\varphi(z)$ over the disk $D(z_0, \varepsilon)$. Condition (2.1) includes the assumption that φ is integrable in some neighborhood of the point z_0. We call $d_\varphi(z_0)$ the **dispersion** of the function φ at the point z_0. We say that a function $\varphi\colon D \to \mathbb{R}$ is of **finite mean oscillation in** D, abbr. $\varphi \in$ FMO(D) or simply $\varphi \in$ **FMO**, if φ has a finite dispersion at every point $z \in D$.

2.3. Remark: Note that, if a function $\varphi\colon D \to \mathbb{R}$ is integrable over $D(z_0, \varepsilon_0) \subset D$, then

$$\fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)|dxdy \le 2 \cdot \overline{\varphi}_\varepsilon(z_0) \tag{2.4}$$

and the left-hand side in (2.4) is continuous in the parameter $\varepsilon \in (0, \varepsilon_0]$ by the absolute continuity of the indefinite integral. Thus, for every $\delta_0 \in (0, \varepsilon_0)$,

$$\sup_{\varepsilon\in[\delta_0,\varepsilon_0]} \fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)|dxdy < \infty. \tag{2.5}$$

If (2.1) holds, then

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)|dxdy < \infty. \tag{2.6}$$

The value of the left-hand side of (2.6) is called the **maximal dispersion** of the function φ in the disk $D(z_0, \varepsilon_0)$.

2.7. PROPOSITION: *If, for some collection of numbers $\varphi_\varepsilon \in \mathbb{R}, \varepsilon \in (0, \varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{D(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon|dxdy < \infty, \tag{2.8}$$

then φ is of finite mean oscillation at z_0.

Proof: Indeed, by the triangle inequality,

$$\begin{aligned}\fint_{D(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)|dxdy &\le \fint_{D(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon|dxdy + |\varphi_\varepsilon - \overline{\varphi}_\varepsilon(z_0)| \\ &\le 2 \cdot \fint_{D(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon|dxdy. \qquad \blacksquare\end{aligned}$$

2.9. COROLLARY: *If, for a point $z_0 \in D$,*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{D(z_0,\varepsilon)} |\varphi(z)|dxdy < \infty, \tag{2.10}$$

then φ has finite mean oscillation at z_0.

2.11. Remark: Clearly BMO $\subset$ FMO. The example given in Section 6 shows that the inclusion is proper. Note that the function $\varphi(z) = \log(1/|z|)$ belongs to BMO in the unit disk Δ (see, e.g., [RR], p. 5) and hence also to FMO. However,

$\overline{\varphi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that the condition (2.8) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0.

Recall that a point $z_0 \in D$ is a **Lebesgue point** of a function $\varphi\colon D \to \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon \to 0} \fint_{D(z_0,\varepsilon)} |\varphi(z) - \varphi(z_0)| dxdy = 0. \tag{2.12}$$

It is known that, for every function $\varphi \in L^1(D)$, almost every point in D is a Lebesgue point. We thus have the following corollary.

2.13. Corollary: *Every function $\varphi\colon D \to \mathbb{R}$, which is locally integrable, has a finite mean oscillation at almost every point in D.*

Below we use the following notation:

$$A(\varepsilon, \varepsilon_0) = \{z \in \mathbb{C}\colon \varepsilon < |z| < \varepsilon_0\}. \tag{2.14}$$

2.15. Lemma: *Let $\varphi\colon D \to \mathbb{R}$ be a nonnegative function with finite mean oscillation at $0 \in D$ and integrable in $D(e^{-1}) \subset D$. Then, for $\varepsilon \in (0, e^{-e})$,*

$$\int_{A(\varepsilon, e^{-1})} \frac{\varphi(z)dxdy}{(|z| \log \frac{1}{|z|})^2} \le C \cdot \log \log \frac{1}{\varepsilon} \tag{2.16}$$

where

$$C = 2\pi(2\varphi_0 + 3e^2 d_0), \tag{2.17}$$

φ_0 is the mean value of φ over the disk $D(e^{-1})$ and d_0 is the maximal dispersion of φ in the disk $D(e^{-1})$.

The lemma plays a central role in the proof of equicontinuity. Versions of this lemma were first established for BMO functions in [RSY_2] and then for FMO functions in [IR_1]. An n-dimensional version of the lemma for BMO functions was established in [MRSY]. The proof here is similar to the one in [RSY_1] and is presented for the sake of completeness.

Proof: Let $0 < \varepsilon < e^{-e}$, $\varepsilon_k = e^{-k}$, $A_k = \{z \in \mathbb{C}^n\colon \varepsilon_{k+1} \le |z| < \varepsilon_k\}$, $D_k = D(\varepsilon_k)$ and let φ_k be the mean value of $\varphi(z)$ over $D_k, k = 1, 2 \ldots$. Take a natural number N such that $\varepsilon \in [\varepsilon_{N+1}, \varepsilon_N)$. Then $A(\varepsilon, e^{-1}) \subset A(\varepsilon) = \bigcup_{k=1}^N A_k$ and

$$\eta(\varepsilon) = \int_{A(\varepsilon)} \varphi(z)\alpha(|z|)dxdy \le |S_1| + S_2,$$

where

$$\alpha(t) = (t \log 1/t)^{-2},$$

$$S_1(\varepsilon) = \sum_{k=1}^{N} \int_{A_k} (\varphi(z) - \varphi_k)\alpha(|z|)dxdy$$

and

$$S_2(\varepsilon) = \sum_{k=1}^{N} \varphi_k \int_{A_k} \alpha(|z|)dxdy.$$

Since $A_k \subset D_k$, $|z|^{-2} \leq \pi e^2/|D_k|$ for $z \in A_k$ and $\log(1/|z|) > k$ in A_k, then

$$|S_1| \leq \pi e^2 d_0 \sum_{k=1}^{N} \frac{1}{k^2} < 2\pi e^2 d_0$$

because

$$\sum_{k=2}^{\infty} \frac{1}{k^2} < \int_1^{\infty} \frac{dt}{t^2} = 1.$$

Now,

$$\int_{A_k} \alpha(|z|)dxdy \leq \frac{1}{k^2} \int_{A_k} \frac{dxdy}{|z|^2} = \frac{2\pi}{k^2}.$$

Moreover,

$$\begin{aligned} |\varphi_k - \varphi_{k-1}| &= \frac{1}{|D_k|} \left| \int_{D_k} (\varphi(z) - \varphi_{k-1})dxdy \right| \\ &\leq \frac{e^2}{|D_{k-1}|} \int_{D_{k-1}} |\varphi(z) - \varphi_{k-1}|dxdy \leq e^2 d_0 \end{aligned}$$

and, by the triangle inequality, for $k \geq 2$,

$$\varphi_k = |\varphi_k| \leq \varphi_1 + \sum_{l=2}^{k} |\varphi_l - \varphi_{l-1}| \leq \varphi_1 + ke^2 d_0 = \varphi_0 + ke^2 d_0.$$

Hence

$$S_2 = |S_2| \leq 2\pi \sum_{k=1}^{N} \frac{\varphi_k}{k^2} \leq 4\pi\varphi_0 + 2\pi e^2 d_0 \sum_{k=1}^{N} \frac{1}{k}.$$

But

$$\sum_{k=2}^{N} \frac{1}{k} < \int_1^{N} \frac{dt}{t} = \log N$$

and, for $\varepsilon < \varepsilon_N$,

$$N = \log \frac{1}{\varepsilon_N} < \log \frac{1}{\varepsilon}.$$

Consequently,

$$\sum_{k=1}^{N} \frac{1}{k} < 1 + \log\log\frac{1}{\varepsilon},$$

and thus for $\varepsilon \in (0, e^{-e})$,

$$\eta(\varepsilon) \leq 2\pi\Big(e^2 d_0 + 2\cdot\frac{e^2 d_0 + \varphi_0}{\log\log\frac{1}{\varepsilon}}\Big)\cdot\log\log\frac{1}{\varepsilon} \leq C\cdot\log\log\frac{1}{\varepsilon},$$

which completes the proof. ∎

2.18. Remark: The concept of finite mean oscillation can be extended to infinity in the standard way. Namely, given a domain D in the extended complex plane $\overline{\mathbb{C}}$, $\infty \in D$, and a function $\varphi\colon D \to \mathbb{R}$, we say that φ has finite mean oscillation at ∞ if the function $\varphi^*(z) = \varphi(1/\overline{z})$ has finite mean oscillation at 0. Clearly, by the change of variables $z \mapsto 1/\overline{z}$, the latter is equivalent to the condition

$$\int_{|z|\geq R} |\varphi(z) - \overline{\varphi}_R|\frac{dxdy}{|z|^4} = O\Big(\frac{1}{R^2}\Big), \tag{2.19}$$

where

$$\overline{\varphi}_R = \int_{|z|\geq R} \varphi(z)\frac{dxdy}{|z|^4}. \tag{2.20}$$

3. Estimate of distortion

For points $a, b \in \mathbb{C}$ and a set $E \subset \overline{\mathbb{C}}$ the **chordal distance** and **chordal diameter** will be denoted by $s(a,b)$ and $\delta(E)$, respectively. Given a domain $D \subset \mathbb{C}$, a measurable function $Q\colon D \to [1,\infty]$ and a number $\Delta > 0$, let $\mathfrak{F}_Q^\Delta$ denote the class of all qc, i.e. quasiconformal, mappings $f\colon D \to \overline{\mathbb{C}}$ such that

$$\delta(\overline{\mathbb{C}} \setminus f(D)) \geq \Delta \tag{3.1}$$

and such that

$$K_f(z) \leq Q(z) \quad \text{a.e. in } D. \tag{3.2}$$

This means, in particular, that f is sense-preserving ACL homeomorphisms, and in addition to (3.2), $K_f \in L^\infty(D)$.

The main tool in establishing distortion estimates for mappings in $\mathfrak{F}_Q^\Delta$ are modulus inequalities for path families Γ in $\mathbb{C}$. Recall that the **modulus** of a path family Γ is defined by

$$M(\Gamma) = \inf_{\rho\in\operatorname{adm}\Gamma} \int_{\mathbb{C}} \rho^2(z)dxdy, \tag{3.3}$$

where $\rho\colon \overline{\mathbb{C}} \to [0,\infty]$ is admissible, $\rho \in \operatorname{adm}\Gamma$, if ρ is a Borel function and

$$\int_\gamma \rho(z)|dz| \geq 1 \tag{3.4}$$

for every $\gamma \in \Gamma$.

The modulus $M(\Gamma)$ is invariant under conformal mappings and quasi-invariant under qc mappings. Namely, if $f\colon D \to \mathbb{C}$ is qc with $K_f(z) \leq K < \infty$ a.e., then

$$M(f\Gamma) \leq KM(\Gamma)$$

for every path family Γ in D. This inequality can be refined as follows. If $f\colon D \to \overline{\mathbb{C}}$ is qc and, in addition, satisfies (3.2) for a given L^1_{loc} function $Q\colon D \to [1,\infty]$, then

$$M(f\Gamma) \leq \int_D\int Q(z)\cdot \rho^2(z)dxdy \tag{3.5}$$

holds for every path family Γ in $\overline{\mathbb{C}}$ and every $\rho \in \operatorname{adm}\Gamma$; see [LV], p. 221.

Given a domain D and two sets E and F in $\overline{\mathbb{C}}$, let $\Gamma(E,F,D)$ denote the family of all paths $\gamma\colon [a,b] \to \overline{\mathbb{C}}$ which join E and F in D, i.e., $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. If $D = \overline{\mathbb{C}}$, we set $\Gamma(E,F) = \Gamma(E,F,\overline{\mathbb{C}})$.

Given a ring $R = R(C_1, C_2)$, i.e. a doubly connected domain R in $\overline{\mathbb{C}}$ with C_1 and C_2 being the connected components of $\overline{\mathbb{C}} \setminus R$, the **capacity** of R can be defined by

$$\operatorname{cap} R(C_1,C_2) = M(\Gamma(C_1,C_2,R)); \tag{3.6}$$

see, e.g., [Ge] and [Zi]. Note that $M(\Gamma(C_1,C_2,R)) = M(\Gamma(C_1,C_2))$.

For $t > 1$, the **Teichmüller ring** $R_T(t)$ is defined by

$$R_T(t) = \overline{\mathbb{C}} \setminus ([-1,0] \cup [t,\infty]). \tag{3.7}$$

By a well-known lemma of Gehring (see, e.g., 7.37 in [Vu] or [Ge]),

$$\operatorname{cap} R(C_1,C_2) \geq \operatorname{cap} R_T\Big(\frac{1}{\delta(C_1)\delta(C_2)}\Big) \tag{3.8}$$

where δ is the spherical diameter. It is also known (see, e.g., (7.19) and (7.22) in [Vu]) that

$$\operatorname{cap} R_T(t) = \frac{2\pi}{\log \Phi(t)}, \tag{3.9}$$

where the function Φ has the following simple estimate:

$$t+1 \le \Phi(t) \le 16\cdot(t+1), \quad t>1. \tag{3.10}$$

Hence by (3.6)–(3.10),

$$M(\Gamma(E,F)) \ge \frac{2\pi}{\log\frac{32}{\delta(E)\delta(F)}} \tag{3.11}$$

for every continua E and F in $\overline{\mathbb{C}}$.

3.12. Lemma: *Let D be a domain in $\mathbb{C}$ with $\overline{D(1/e)}\subset D$ and $f\colon D\to\mathbb{C}$ a qc mapping. If $f\in f\in\mathfrak{F}_Q^\Delta$ for some $\Delta>0$ and some function $Q\colon D(1/e)\to[1,\infty]$ which is integrable in $D(1/e)$ and which is of finite mean oscillation at 0, then*

$$s(f(z),f(0)) \le \alpha_0\cdot\left(\log\frac{1}{|z|}\right)^{-\beta_0} \tag{3.13}$$

for every point $z\in D(e^{-e})$, where

$$\alpha_0 = 32/\Delta, \tag{3.14}$$

$$\beta_0 = (2q_0+3e^2d_0)^{-1}, \tag{3.15}$$

q_0 is the mean value of $Q(z)$ in $D(1/e)$ and d_0 is the maximal dispersion of $Q(z)$ in $D(1/e)$.

Proof: Let Γ_ε denote the family of all paths joining the circles

$$S_\varepsilon=\{z\in\mathbb{C}\colon |z|=\varepsilon\} \quad\text{and}\quad S_0=\{z\in\mathbb{C}\colon |z|=e^{-1}\}$$

in the ring $A_\varepsilon=\{z\in\mathbb{C}\colon \varepsilon<|z|<e^{-1}\}$. Then the function

$$\rho_\varepsilon(z)=\begin{cases} a_\varepsilon/|z|\log(1/|z|), & \text{if } z\in A_\varepsilon,\\ 0, & \text{if } z\in\mathbb{C}\backslash A_\varepsilon,\end{cases} \tag{3.16}$$

where

$$a_\varepsilon=\left(\log\log\frac{1}{\varepsilon}\right)^{-1}, \tag{3.17}$$

is admissible for Γ_ε, and hence by (3.5),

$$M(f\Gamma_\varepsilon) \le \int_D Q(z)\cdot\rho_\varepsilon^2(|z|)dxdy. \tag{3.18}$$

Now $Q\in$ FMO at 0, and thus by Lemma 2.15, (3.16) and (3.17),

$$\int_D Q(z)\cdot\rho_\varepsilon^2(|z|)dxdy \le C\cdot a_\varepsilon \tag{3.19}$$

and hence

$$M(f\Gamma_\varepsilon) \le \frac{C}{\log\log(1/\varepsilon)}, \tag{3.20}$$

where the constant C is as in Lemma 2.15.

Note that $\overline{\mathbb{C}}\backslash fA_\varepsilon$ has exactly two components, because f is a homeomorphism and A_ε is a doubly connected domain. Denote by Γ_ε^* the family of all paths in $\overline{\mathbb{C}}$ joining the sets fS_ε and fS_0. Then

$$M(\Gamma_\varepsilon^*) = M(f\Gamma_\varepsilon). \tag{3.21}$$

Indeed, on one hand $f\Gamma_\varepsilon \subset \Gamma_\varepsilon^*$ and hence $M(f\Gamma_\varepsilon) \le M(\Gamma_\varepsilon^*)$, and on the other hand $f\Gamma_\varepsilon < \Gamma_\varepsilon^*$, i.e. every path in Γ_ε^* contains a subpath which belongs to $f\Gamma_\varepsilon$, and hence $M(f\Gamma_\varepsilon) \ge M(\Gamma_\varepsilon^*)$; see, e.g., Theorem 1(c) in [Fu]. Finally, by choosing $\varepsilon = |z|$ and invoking (3.11), we obtain (3.13). ∎

3.22. Corollary: *Let D be a domain in $\mathbb{C}$, $D(z_0, r_0)$ a disk in $D, f\colon D \to \overline{\mathbb{C}}$ a qc mapping which belongs to $\mathfrak{F}_Q^\Delta$ for some $\Delta > 0$ and some function $Q\colon D(z_0,r_0) \to [1,\infty]$ which is integrable in $D(z_0,r_0)$. If $Q(z)$ is of finite mean oscillation at z_0, then*

$$s(f(z), f(z_0)) \le \alpha_0 \cdot \left(\log \frac{er_0}{|z-z_0|}\right)^{-\beta_0} \tag{3.23}$$

for every $z \in D(z_0, e^{1-e}r_0)$, where α_0 and β_0 are as in (3.14) and (3.15), q_0 is the mean value of the function $Q(z)$ over the disk $D(z_0,r_0)$ and d_0 is the maximal dispersion of $Q(z)$ in $D(z_0,r_0)$.

Indeed, the mean value and the dispersion of a function over disks are invariant under linear transformations of the independent variable, thus (3.23) follows from Lemma 3.12 by applying the transformation $z \mapsto (z-z_0)/er_0$.

4. Existence theorems

Based on the distortion estimates of Section 3, we obtain by a standard approximation process the following existence theorem.

4.1. Theorem: *Let D be a domain in $\mathbb{C}$ and $\mu\colon D \to \mathbb{C}$ a measurable function with $|\mu(z)| < 1$ a.e. If*

$$K_\mu(z) \le Q(z) \quad \text{a.e. in } D \tag{4.2}$$

for some FMO function $Q\colon D \to [1,\infty]$, then the Beltrami equation (1.1) has a homeomorphic $W^{1,1}_{loc}$ solution $f_\mu\colon D \to \mathbb{C}$ with $f_\mu^{-1} \in W^{1,2}_{loc}$.

Proof: Fix points z_1 and z_2 in D. For $n \in \mathbb{N}$, define $\mu_n\colon D \to \mathbb{C}$ by letting $\mu_n(z) = \mu(z)$ if $|\mu(z)| \le 1 - 1/n$ and 0 otherwise. Then $\|\mu_n\|_\infty < 1$, and thus, by the classical existence theorem, the Beltrami equation (1.1) with μ_n instead of μ has a homeomorphic ACL solution $f_n\colon D \to \mathbb{C}$ which fixes z_1 and z_2; see, e.g., [Ah] and [LV]. By Corollary 3.22 the sequence f_n is equicontinuous, and hence by the Arzela–Ascoli theorem (see, e.g., [Du], p. 267, and [DS], p. 382) it has a subsequence, denoted again by f_n, which converges locally uniformly to some nonconstant mapping f in D. Then f is $Q(z)$-qc and satisfies (1.1) a.e.; see Theorem 3.1 and Corollary 5.12 in [RSY$_1$] on the convergence of $Q(z)$-qc mappings. It follows, in particular, that f is an ACL homeomorphic solution of (1.1). Since $Q \in FMO$, $Q \in L^1_{loc}$, and hence $K_\mu \in L^1_{loc}$. Therefore, by a standard arguments f is a $W^{1,1}_{loc}$ solution. ∎

Next, note that the local convergence $f_n \to f$ is equivalent to its continuous convergence, i.e., $f_n(z_n) \to f(z_0)$ if $z_n \to z_0$; see [Du], p. 268. Since f is injective, it follows that $g_n = f_n^{-1} \to f^{-1} = g$ continuously, and hence locally uniformly. By direct computation we obtain that for large n,

$$\int_B |\partial g_n|^2 du dv = \int_{g_n(B)} \frac{dxdy}{1-|\mu_n(z)|^2} \le \int_{B^*} Q(z) dxdy < \infty, \tag{4.3}$$

where B^* and B are relatively compact domains in D and in $f(D)$, respectively, such that $g(\bar{B}) \subset B^*$. The change of variables is allowed since g_n and f_n are qc and hence in $W^{1,2}_{loc}$. The relation (4.3) implies that the sequence g_n is bounded in $W^{1,2}(B)$, and hence $f^{-1} \in W^{1,2}_{loc}(f(D))$; see, e.g., [Re], p. 319.

4.4. COROLLARY: *f_μ^{-1} is locally absolutely continuous and preserves nulls sets, and f_μ is regular, i.e., differentiable with $J_{f_\mu}(z) > 0$ a.e.*

Indeed, the assertion about f_μ^{-1} follows from the fact that $f_\mu^{-1} \in \mathrm{W}^{1,2}_{loc}$; see [LV], pp. 131 and 150. As an ACL mapping, f_μ has a.e. partial derivatives and hence by [GL] it has a total differential a.e. Let E denote the set of points of D where f_μ is differentiable and $J_{f_\mu}(z) = 0$, and suppose that $|E| > 0$. Then $|f_\mu(E)| > 0$, since $E = f_\mu^{-1}(f_\mu(E))$ and f_μ^{-1} preserves null sets. Clearly f_μ^{-1} is not differentiable at any point of $f_\mu(E)$, contradicting the fact that f_μ^{-1} is differentiable a.e.

4.5. COROLLARY: *If $K_\mu(z) \le Q(z)$ a.e. and every point $z \in D$ is a Lebesgue point for $Q(z)$, then the Beltrami equation (1.1) has a homeomorphic $W^{1,1}_{loc}$ solution f_μ with $f_\mu^{-1} \in W^{1,2}_{loc}$.*

4.6. COROLLARY: *If, for every point $z_0 \in D$,*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{D(z_0,\varepsilon)} K_\mu(z)dxdy < \infty, \tag{4.7}$$

then the Beltrami equation (1.1) has a homeomorphic $W^{1,1}_{loc}$ solution f_μ with $J_{f_\mu}(z) > 0$ a.e.

4.8. Remark: (1) Note that if $f: \mathbb{C} \to \mathbb{C}$ is $Q(z)$-qc with $Q \in BMO$, then f is surjective and extends to a BMO-qc mapping of $\overline{\mathbb{C}}$ onto itself, because isolated singularities are removable for BMO functions; see, e.g., [RSY$_2$]. However, a $Q(z)$-qc mapping f need not be surjective if the condition $Q \in BMO$ is replaced by the weaker condition $Q \in FMO$, for instance, the dilatation of every diffeomorphism of $\mathbb{C}$ onto the unit disk Δ is continuous and hence belongs to FMO yet $f: \mathbb{C} \to \mathbb{C}$ is not surjective. Note that f^{-1} is a diffeomorphism of Δ onto $\mathbb{C}$ and its dilatation is also in FMO.

(2) In view of Lemma 3.12, Theorem 4.1 extends to the case where $\infty \in D \subset \overline{\mathbb{C}}$ if the condition that $Q(z)$ has finite mean oscillation at ∞ is added; see Remark 2.18. In this case there exists a homeomorphic $W^{1,1}_{loc}$ solution $f = f_\mu$ in D with $f(\infty) = \infty$ and $f_\mu^{-1} \in W^{1,2}_{loc}$. Here $f \in W^{1,1}_{loc}$ in D means that $f \in W^{1,1}_{loc}$ in $D \setminus \{\infty\}$ and that $f^*(z) = 1/\overline{f(1/\overline{z})}$ belongs to $W^{1,1}$ in a neighborhood of 0. The statement $f^{-1} \in W^{1,2}_{loc}$ has a similar meaning. Consequently, if the condition (4.7) holds at every point $z_0 \in D \setminus \{\infty\}$ and if

$$\int_{|z|>R} K_\mu(z)\frac{dxdy}{|z|^4} = O\Big(\frac{1}{R^2}\Big) \quad \text{as } R \to \infty, \tag{4.9}$$

then the Beltrami equation (1.1) has a homeomorphic $W^{1,1}_{loc}$ solution f_μ with $f_\mu^{-1} \in W^{1,2}_{loc}$.

(3) In view of the dilatation estimate in Lemma 3.12 and Corollary 3.22, the condition (4.2) in Theorem 4.1 can be localized, yielding the following corollaries.

4.10. COROLLARY: *Let D be a domain in $\mathbb{C}$ and $\mu: D \to \mathbb{C}$ a measurable function with $|\mu(z)| < 1$ a.e. If for every point $z_0 \in D$, there exist a disk $D(z_0, r_0) \subset D$ and a function $Q_{z_0}: D(z_0,r_0) \to [1,\infty]$ which is of finite mean oscillation at z_0 such that $K_\mu(z) \le Q_{z_0}(z)$ for a.e. $z \in D(z_0,r_0)$, then the*

Beltrami equation (1.1) has a homeomorphic $W^{1,1}_{loc}$ solution f_μ: $D \to \mathbb{C}$ with $f_\mu^{-1} \in W^{1,2}_{loc}$.

4.11. COROLLARY: *Let μ: $\mathbb{C} \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. If*

$$K_\mu(z) = O\Big(\log \frac{1}{|z - z_0|}\Big) \quad \text{as } z \to z_0 \tag{4.12}$$

for every point $z_0 \in \mathbb{C}$ and

$$K_\mu(z) = O(\log |z|) \quad \text{as } z \to \infty, \tag{4.13}$$

then the Beltrami equation (1.1) has a homeomorphic $W^{1,1}_{loc}$ solution f_μ: $\mathbb{C} \to \mathbb{C}$ with $f(\mathbb{C}) = \mathbb{C}$ such that $f_\mu^{-1} \in W^{1,2}_{loc}$.

5. Representation, factorization and uniqueness theorems

In Section 4, we established the existence of a homeomorphic solution f_μ of the Beltrami equation (1.1) in a domain $D \subset \mathbb{C}$ in the case where $K_\mu(z) \leq Q(z)$ a.e. in D for some FMO function Q: $D \to [1, \infty]$. We now show that, for such a coefficient μ, the specific solution f_μ which is constructed in Theorem 4.1 generates all $W^{1,2}_{loc}$ solutions by composition with analytic functions. The proof uses merely the fact that $K_\mu \in L^1_{loc}$ and $f_\mu^{-1} \in W^{1,2}_{loc}$ and the main argument is as in Brakalova and Jenkins [BJ], p. 86. It should be noted that by a recent result of Hencl and Koskela [HK], $f^{-1} \in W^{1,2}_{loc}$ holds for any $W^{1,1}_{loc}$-homeomorphism with $K_f \in L^1_{loc}$.

5.1. THEOREM: *Let D be a domain in $\mathbb{C}$ and μ: $D \to \mathbb{C}$ a measurable function with $|\mu(z)| < 1$ a.e. and*

$$K_\mu(z) \leq Q(z) \quad \text{a.e. in } D, \tag{5.2}$$

for some FMO function Q: $D \to [1, \infty]$. Then every $W^{1,2}_{loc}$ solution g of the Beltrami equation has the representation

$$g = h \circ f_\mu, \tag{5.3}$$

where f_μ is the solution given in Theorem 4.1 and h is holomorphic in $f_\mu(D)$.

Proof: Let $\varphi = f_\mu^{-1}$ and $h = g \circ \varphi$. Since $g \in W^{1,2}_{loc}$ and $\varphi \in W^{1,2}_{loc}$, it follows that $h \in \mathrm{W}^{1,1}_{loc}(f(D))$; see [LV], p. 151. Thus, by Weyl's lemma (see, e.g., [Ah],

p. 33) it suffices to show that $\overline{\partial} h = 0$ a.e. in $f_\mu(D)$. Let E denote the set of points z in D where either f_μ or g do not satisfy (1.1) or $J_{f_\mu} = 0$. A direct computation (cf. [Ah], p. 9) shows that $\overline{\partial} h = 0$ in $f_\mu(D) \setminus f_\mu(E)$. Moreover, $\varphi \in W^{1,2}_{loc}$ admits the change of variables (see, e.g., [LV], pp. 121, 128–130 and 150)

$$\int_{f_\mu(E)}\int |\partial\varphi|^2 dudv = \int_{f_\mu(E)}\int J_\varphi(w)\frac{dudv}{1-|\mu(\varphi(w))|^2} = \int_E\int \frac{dxdy}{1-|\mu(z)|^2} = 0,$$

which implies that $|\partial\varphi| = 0$ a.e. in $f_\mu(E)$. Also, $|\overline{\partial}\varphi| \leq |\partial\varphi|$ a.e. and

$$\overline{\partial} h = \overline{\partial}\varphi \cdot \partial g \circ \varphi + \overline{\partial\varphi} \cdot \overline{\partial} g \circ \varphi,$$

hence $|\overline{\partial} h| = 0$ a.e. in $f_\mu(E)$, and thus $\overline{\partial} h = 0$ a.e. in $f_\mu(D)$. Consequently, h is holomorphic in $f_\mu(D)$ and (5.3) holds. ∎

5.4. COROLLARY: *Let μ be as in Theorem 5.1. Then every nonconstant $W^{1,2}_{loc}$ solution g of (1.1) is open, discrete and regular a.e., i.e., g is differentiable and $J_g(z) \neq 0$ a.e.*

Given $K_\mu \leq Q$ a.e., $Q \in FMO$, it is not clear whether an ACL homeomorphic solution of (1.1) is unique up to a composition with a conformal mapping, namely whether, for any two ACL homeomorphic solutions f_1 and f_2 of (1.1), $f_2 \circ f_1^{-1}$ is conformal. However, as a consequence of Theorem 5.1 we have an affirmative answer in the special case when f_1 and f_2 belong to $W^{1,2}_{loc}$ as stated in the following corollary. Another type of condition for the uniqueness of a homeomorphic ACL solution can be obtained by imposing certain restrictions on the "size" of the singular set of μ. This will be done in Theorem 5.17.

5.5. COROLLARY: *Let $\mu\colon D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. in D and such that (5.2) holds for some FMO function $Q\colon D \to [1,\infty]$. If f_1 and f_2 are homeomorphic $W^{1,2}_{loc}$ solutions of (1.1) in D, then $f_2 \circ f_1^{-1}$ is conformal.*

5.6. COROLLARY: *Let $\mu\colon D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. in D. If*

$$\overline{\lim_{\varepsilon\to 0}} \fint_{D(z_0,\varepsilon)} K_\mu(z)dxdy < \infty, \tag{5.7}$$

at every point $z_0 \in D$, then (1.1) has a homeomorphic solution f_μ and every $W^{1,2}_{loc}$ solution g has the representation (5.3).

5.8. COROLLARY: *Let $\mu\colon D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. in D. If*

$$K_\mu(z) = O\Big(\log \frac{1}{|z - z_0|}\Big) \tag{5.9}$$

as $z \to z_0$ at every point $z_0 \in D$, then (1.1) has a homeomorphic solution f_μ and every $W^{1,2}_{loc}$ solution g has the representation (5.3).

Iwaniec and Šverák (Theorem 1 in [IS]) showed that if $g\colon D \to \mathbb{C}$ belongs to $W^{1,2}_{loc}$, $J(z,g) \geq 0$ a.e. and $K_g \in L^1_{loc}$, then there exist a homeomorphism $f\colon D \to \mathbb{C}$ such that $f^{-1} \in W^{1,2}_{loc}$ and a holomorphic function $h\colon f(\mathbb{C}) \to \mathbb{C}$ such that $g = h \circ f$. A simple argument for this statement in the special case where $K_\mu \in L^1_{loc}$ is replaced by the more restrictive condition that $K_\mu(z) \leq Q(z)$ a.e. for some FMO function Q appears in the proof of Theorem 5.1. It should be noted that Iwaniec and Martin have constructed μ's with $K_\mu \in L^1_{loc}$ and corresponding ACL solutions which belong to all $W^{1,p}_{loc}, p < 2$, which are not open and discrete, and thus are not generated by a homeomorphic solution in the sense of (5.3); see, e.g., [IM].

By Stoilow's factorization theorem, it is easy to obtain the following conclusion.

5.10. PROPOSITION: *Let $\mu\colon D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. such that*

$$K_\mu(z) \in L^1_{loc}. \tag{5.11}$$

Then every discrete and open ACL solution g of the Beltrami equation (1.1) has the representation $g = h \circ f$, where f is a homeomorphic $W^{1,1}_{loc}$ solution of (1.1) and h is a holomorphic function in $f(D)$.

5.12. Remark: As a consequence of the proposition we obtain that if $K_\mu \in L^1_{loc}$, then either the Beltrami equation (1.1) has a homeomorphic $W^{1,1}_{loc}$ solution or has no discrete and open ACL solution.

We end this section with a uniqueness theorem involving a condition on the singular set S_μ of μ which is defined below.

Let (X,d) be a metric space and let $H = \{h_x(r)\}_{x\in X}$ be a family of positive functions h_x given on $(0,\rho_x)$, $\rho_x > 0$, such that $h_x(r) \to 0$ as $r \to 0$. Let

$$\Lambda^\rho_H(X) = \inf \Sigma h_{x_k}(r_k), \tag{5.13}$$

where the infimum is taken over all finite collections of $x_k \in X$ and $r_k \in (0, \rho)$ such that the balls

$$B(x_k, r_k) = \{x \in X\colon d(x, x_k) < r_k\} \tag{5.14}$$

cover X. The limit

$$\Lambda_H(X) = \lim_{\rho \to 0} \Lambda_H^\rho(X) \tag{5.15}$$

exists and is called the **H-length** of X. In particular, if $h_x(r) = r$ for all $x \in X$ and $r > 0$, then $\Lambda_H(X)$ is length of X.

The **singular set** S_μ of $\mu\colon D \to \mathbb{C}$ is defined by

$$S_\mu = \{z \in D\colon \lim_{\varepsilon \to 0} \|K_\mu\|_{L^\infty(D(z,\varepsilon))} = \infty\}. \tag{5.16}$$

Obviously, the set S_μ is closed relatively to the domain D.

5.17. THEOREM: *Let μ, Q and f_μ be as in Theorem 4.1. For z in the singular set S_μ of μ and $0 < r < \delta(z) = \operatorname{dist}(z, \partial D)$, let*

$$h_z(r) = \left(\log \frac{\delta(z)}{r}\right)^{-\beta(z)}, \tag{5.18}$$

where $\beta(z) = (2q(z) + 3e^2 d(z))^{-1}$, $q(z)$ is the mean value of $Q(\zeta)$ over $D(z, e^{-1}\delta(z))$ and $d(z)$ is the maximal dispersion of $Q(\zeta)$ in $D(z, e^{-1}\delta(z))$. Let $H = \{h_z(r)\}_{z \in S_\mu}$.

If S_μ is of H-length zero, then for every homeomorphic ACL solution f of the Beltrami equation (1.1) there is a conformal mapping h such that $f = h \circ f_\mu$.

Proof: If $\Lambda_H(S_\mu) = 0$, then $S'_\mu = f_\mu(S_\mu)$ is of length zero by Corollary 3.22 with $r_0 = e^{-1}\delta$. Consequently, S'_μ does not locally disconnect $\mathbb{C}$ (see, e.g., [Vä]) and hence $G = D \setminus S_\mu$ is a domain. The homeomorphisms f and f_μ are locally quasiconformal in the domain G and hence $h = f \circ f_\mu^{-1}$ is conformal in the domain $f_\mu(D) \setminus S'_\mu$. Since S'_μ is of length zero, it is removable for h, i.e., h can be extended to a conformal mapping in $f_\mu(D)$ by the Painlevé theorem; see, e.g., [Be]. ∎

5.19. Remark: In view of Remark 2.3, if

$$\overline{Q}(z_0) = \overline{\lim_{\varepsilon \to 0}} \fint_{D(z_0,\varepsilon)} |Q(z)|\,dxdy < \infty, \tag{5.20}$$

for every point $z_0 \in D$, then one may take $\beta(z) = \gamma / \overline{Q}(z)$ in (5.18) for any $\gamma < 1/(2 + 6e^2)$.

6. Examples

We conclude this paper by constructing a function $\varphi\colon \mathbb{C} \to \mathbb{R}$ which belongs to FMO but not to L^p_{loc} for any $p > 1$, and hence not to BMO_{loc}. In the following examples, $p = 1 + \delta$ with an arbitrarily small $\delta > 0$. Let

$$\varphi(z) = \begin{cases} e^{1/(|z|^2-1)}, & \text{if } |z| < 1, \\ 0, & \text{if } |z| \geq 1. \end{cases} \tag{6.1}$$

Then φ belongs to C_0^∞ and hence to BMO. Consider the function

$$\varphi^*_\delta(z) = \begin{cases} \varphi_k(z), & \text{if } z \in D_k, \\ 0, & \text{if } z \in \mathbb{C} \setminus \cup D_k, \end{cases} \tag{6.2}$$

where $D_k = D(z_k, r_k)$, $z_k = 2^{-k}$, $r_k = 2^{-(1+\delta)k^2}$, $\delta > 0$, and

$$\varphi_k(z) = 2^{2k^2} \varphi\Big(\frac{z - z_k}{r_k}\Big), \quad z \in D_k, \ k = 2, 3, \ldots. \tag{6.3}$$

Then φ^*_δ is smooth in $\mathbb{C} \setminus \{0\}$ and thus belongs to $\mathrm{BMO}_{loc}(\mathbb{C} \setminus \{0\})$, and hence to $\mathrm{FMO}(\mathbb{C} \setminus \{0\})$.

Now

$$\int_{D_k} \varphi_k(z) dxdy = 2^{-2\delta k^2} \int_{\mathbb{C}} \varphi(z) dxdy. \tag{6.4}$$

Hence

$$\overline{\lim_{\varepsilon \to 0}} \fint_{D(\varepsilon)} \varphi^*_\delta(z) dxdy < \infty. \tag{6.5}$$

Thus, by Corollary 2.9, $\varphi \in$ FMO.

Indeed, by setting

$$K = K(\varepsilon) = \big[\log_2 \frac{1}{\varepsilon}\big] \leq \log_2 \frac{1}{\varepsilon}, \tag{6.6}$$

where $[A]$ denotes the integral part of the number A, we have

$$J = \fint_{D(\varepsilon)} \varphi^*_\delta(z) dxdy \leq I \cdot \sum_{k=K}^{\infty} 2^{-2\delta k^2} / \pi 2^{-2(K+1)}, \tag{6.7}$$

where $I = \int_{\mathbb{C}} \varphi(z) dxdy$. If $\delta K > 1$, i.e. $K > 1/\delta$, then

$$\sum_{k=K}^{\infty} 2^{-2\delta k^2} \leq \sum_{k=K}^{\infty} 2^{-2k} = 2^{-2K} \sum_{k=0}^{\infty} \Big(\frac{1}{4}\Big)^k = \frac{4}{3} \cdot 2^{-2K}, \tag{6.8}$$

i.e., $J \leq 16I/3\pi$.

On the other hand,

$$\int_{D_k} \varphi_k^{1+\delta}(z)dxdy = \int_{\mathbb{C}} \varphi^{1+\delta}(z)dxdy, \tag{6.9}$$

and hence $\varphi_\delta^* \notin L^{1+\delta}(U)$ for any neighborhood U of 0.

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